Stochastic process leading to wave equations in dimensions higher than one

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Stochastic processes are proposed whose master equations coincide with classical wave, telegraph, and Klein-Gordon equations. Similar to predecessors based on the Goldstein-Kac telegraph process, the model describes the motion of particles with constant speed and transitions between discreet allowed velocity directions. A new ingredient is that transitions into a given velocity state depend on spatial derivatives of other states populations, rather than on populations themselves. This feature requires the sacrifice of the single-particle character of the model, but allows to imitate the Huygens' principle and to recover wave equations in arbitrary dimensions.

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I. INTRODUCTION

Can a wave, which is in general an essentially dynamical process, be mapped onto a kinematic stochastic model like random walks? This question has a long history and was approached from many different perspectives, ranging from pragmatic (numerical simulation of wave-related phenomena) to fundamental (interpretation of quantum mechanics) with many important incentives in between, such as effects of inertia in heat transfer, light propagation in turbid media, turbulence diffusion, etc. While random walks are natural underlying processes for parabolic equations of diffusion type, the connections between stochastic motion and hyperbolic wave equations are less obvious and often restricted to one spatial dimension (1D). Perhaps the best-known example is the telegraph equation $f_{tt}+\frac{1}{\tau}f_t=c^2\Delta f$, which describes propagation of waves in media with losses (with characteristic dissipation time τ). In 1D the equation can be readily derived from persistent random walk with constant speed and Poissonian velocity reversals (often referred to as the Goldstein-Kac process) [1–3], but the same walk extended to higher dimensions does not evolve according to the telegraph equation [2,4–7]. This feature is generic and inherited in many related problems, in particular of mapping relativistic quantum wave equations onto classical random walks [9–11]. Such mappings are typically designed in 1D, and the extension to higher spatial dimensions requires the formal replacement of the space-variable derivative by the gradient, $\partial/\partial x \to \nabla$. It was noted by many authors that this approach may be inconsistent since in general it is impossible to construct the random walk with desirable properties (governed by a master equation of desirable form) as a mere superposition of independent one-dimensional walks.

As will be discussed below, the difficulty of extending wave-particle isomorphism beyond 1D is not related to stochastic nature of the random walk models, but rather originates from the inability for a *single-particle* motion, neither stochastic nor deterministic, to imitate the Huygens' principle in dimensions higher than one [12]. On the other hand, there are often no reasons to restrict oneself to single-particle models, which cannot reproduce physically relevant negative solutions anyway. In this paper we discuss a model *à la* Goldstein-Kac based on an ensemble of classical particles moving with a fixed speed and subjected to transitions between discreet allowed directions of motion. It is shown that transitions can be chosen in a form which allows to recover wavelike equations for the ensemble's distribution function in any dimension.

II. WAVE EQUATION

As well known, the classical wave equation in 1D $f_{tt}=c^2f_{xx}$ can be mapped onto a totally deterministic kinematic model. Thanks to the factorization $(\partial/\partial t-c\,\partial/\partial x)(\partial/\partial t+c\,\partial/\partial x)f=0$, the general solution can be written as the superposition of the functions $f^{\pm}(x,t)$, which satisfy the equations

$$\frac{\partial f^+}{\partial t}=-c\frac{\partial f^+}{\partial x},\quad \frac{\partial f^-}{\partial t}=c\frac{\partial f^-}{\partial x}, \tag{1}$$

and can be interpreted as the distribution functions for independent particles, or for an ensemble of single particles, moving freely with constant speed c in positive and negative directions, respectively. Equations (1) also can be written in terms the total distribution function $f=f^+ + f^-$ and the current $J=c(f^+-f^-)$,

$$\frac{\partial f}{\partial t}=-\frac{\partial J}{\partial x},\quad \frac{\partial J}{\partial t}=-c^2\frac{\partial f}{\partial x}. \tag{2}$$

One might suggest that for higher dimensions the proper generalization of Eqs. (2) should read

$$\frac{\partial f}{\partial t}=-\vec{\nabla}\cdot\vec{J},\quad \frac{\partial \vec{J}}{\partial t}=-c^2\vec{\nabla}f, \tag{3}$$

which indeed immediately gives the multidimensional wave equation for f,

$$\frac{\partial^2 f}{\partial t^2}=c^2\Delta f, \tag{4}$$

as well as the conservation law for the current vorticity

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$$\frac{\partial}{\partial t}\vec{\nabla}\times\vec{J}=0. \tag{5}$$

However, it is easy to see that the Eqs. (3) do not follow from multidimensional generalization of Eqs. (1) and therefore cannot describe a single-particle motion in $D>1$.

It is instructive to illustrate the last statement explicitly for two spatial dimensions (2D), assuming that particles move with a constant speed c and can be in one of four velocity states $(x,\pm)$, $(y,\pm)$, corresponding to motions along positive (+) or negative (−) directions of the two Cartesian axes. Let $f_x^{\pm}(x,y,t)$ and $f_y^{\pm}(x,y,t)$ be the corresponding distribution functions. (From here on, we use subscripts $\alpha=x,y,z$ only to refer to vector components, not to partial derivatives). For freely moving particles there are no transitions between the states, so the two-dimensional version of Eqs. (1) reads as follows:

$$\frac{\partial f_x^{\pm}}{\partial t}=\mp c\frac{\partial f_x^{\pm}}{\partial x},\quad \frac{\partial f_y^{\pm}}{\partial t}=\mp c\frac{\partial f_y^{\pm}}{\partial y}. \tag{6}$$

Let $f_x=f_x^+ + f_x^-$, $f_y=f_y^+ + f_y^-$, and $J_x(x,y,t)=c(f_x^+ - f_x^-)$ and $J_y(x,y,t)=c(f_y^+ - f_y^-)$ are Cartesian components of the current $\vec{J}=\vec{i}J_x+\vec{j}J_y$. Adding and subtracting Eqs. (6) one obtains

$$\frac{\partial f_x}{\partial t}=-\frac{\partial J_x}{\partial x},\quad \frac{\partial J_x}{\partial t}=-c^2\frac{\partial f_x}{\partial x} \tag{7}$$

for particles moving along x axis, and

$$\frac{\partial f_y}{\partial t}=-\frac{\partial J_y}{\partial y},\quad \frac{\partial J_y}{\partial t}=-c^2\frac{\partial f_y}{\partial y}, \tag{8}$$

for particles moving along y axis. These equations do not form the closed system (3) for the total distribution $f(x,y,t)=f_x+f_y$ and the current $\vec{J}$,

$$\frac{\partial f}{\partial t}=-\vec{\nabla}\cdot\vec{J},$$

$$\frac{\partial \vec{J}}{\partial t}=-c^2\left\{\vec{i}\frac{\partial f_x}{\partial x}+\vec{j}\frac{\partial f_y}{\partial y}\right\}\neq -c^2\vec{\nabla}f. \tag{9}$$

Therefore the two-dimensional wave equation is not recovered.

Let us now generalize Eqs. (6) by allowing transitions between $(x,+)$ and $(x,-)$ states with transition rate $g_x(x,y,t)$, and between $(y,+)$ and $(y,-)$ states with transition rate $g_y(x,y,t)$,

$$\frac{\partial f_x^{\pm}}{\partial t}=\mp c\frac{\partial f_x^{\pm}}{\partial x}\mp g_x,\quad \frac{\partial f_y^{\pm}}{\partial t}=\mp c\frac{\partial f_y^{\pm}}{\partial y}\mp g_y. \tag{10}$$

(Note that no transitions are still allowed between x and y states.) The presence of transitions does not violate the conservation of the number of particles, so the continuity equation $\partial f/\partial t=-\vec{\nabla}\cdot\vec{J}$ still holds, while the equation for the current now reads

$$\frac{\partial \vec{J}}{\partial t}=-c^2\left\{\vec{i}\left(\frac{\partial f_x}{\partial x}+\frac{2}{c}g_x\right)+\vec{j}\left(\frac{\partial f_y}{\partial y}+\frac{2}{c}g_y\right)\right\}. \tag{11}$$

By choosing transition rates in the form

$$g_x=\frac{c}{2}\frac{\partial f_y}{\partial x},\quad g_y=\frac{c}{2}\frac{\partial f_x}{\partial y} \tag{12}$$

the right-hand side of Eq. (11) is completed to the gradient, $\partial\vec{J}/\partial t=-c^2\vec{\nabla}f$. Thus the system (3) is recovered, and therefore the total distribution f is governed by the two-dimensional wave Eq. (4).

With transition rates [Eq. (12)], the partial motions along x and y axes become statistically coupled even though there are no direct transitions $(x,\pm)\leftrightarrow(y,\pm)$. Note that an equation for the particle distribution in a given state, say $(x,+)$,

$$\frac{\partial f_x^+}{\partial t}=-c\frac{\partial f_x^+}{\partial x}-\frac{c}{2}\frac{\partial f_y}{\partial x}, \tag{13}$$

has a form of the conservation law $\partial f_x^+/\partial t=-\partial J_x^+/\partial x$, where the flux $J_x^+=c(f_x^+ + \frac{1}{2}f_y)$ is determined not only by the density of the given state f_x^+ but also by local particles in other velocity states f_y^+ and f_y^-. This is reminiscent of the Huygens' principle, according to which every point of a wave front propagating with a speed c is the source of secondary waves that spread out in all directions with the same speed c. The resemblance is achieved at the expense of losing the single-particle status of the original 1D model. It is easy to see that with transition rates [Eq. (12)], proportional to derivatives of state populations, the partial distributions $f_x^{\pm}$ and $f_y^{\pm}$ are not positively defined. Then Eqs. (10) cannot describe a single-particle motion, but should be interpreted as the equations for corresponding perturbations in an ensemble of particles.

The generalization of the scheme for 3D is straightforward and reads

$$\frac{\partial f_\alpha^{\pm}}{\partial t}=\mp c\frac{\partial f_\alpha^{\pm}}{\partial \alpha}\mp g_\alpha,\quad \alpha=x,y,z \tag{14}$$

with

$$g_x=\frac{c}{2}\left(\frac{\partial f_y}{\partial x}+\frac{\partial f_z}{\partial x}\right) \tag{15}$$

and similar expressions for g_y and g_z. The Eqs. (14) and (15) lead to the system (3) and therefore to the wave equation in 3D.

A trick of coupling of partial motions with transition rates in the form (12) or (15) is quite generic and can be applied to derive other multidimensional wavelike equations. Two more examples are presented below.

III. TELEGRAPH EQUATION

A hybrid of the wave and diffusion equations, the telegraph equation has a form

$$\frac{\partial^2 f}{\partial t^2}+\frac{1}{\tau}\frac{\partial f}{\partial t}=c^2\Delta f \tag{16}$$

and describes the propagation of waves in media with losses. It is also often used as an approximation to treat diffusion

processes beyond the overdamped limit and in a turbulent medium [1,2,8]. A one-dimensional version of Eq. (16) can be derived as a master equation for the Goldstein-Kac stochastic process—a dichotomous persistent random walk in which a particle moves in 1D with the velocity fluctuating between c and $-c$. Using the same notations as in the previous section, let $f^+(x,t)$ and $f^-(x,t)$ be the probability density for the particle moving to the right and to the left, respectively. Reversals of velocity are Poisson distributed and occurring with the rate $1/2\tau$. The processes is described by equations

$$\frac{\partial f^+}{\partial t} = -c\frac{\partial f^+}{\partial x} - \frac{1}{2\tau}(f^+ - f^-),$$

$$\frac{\partial f^-}{\partial t} = +c\frac{\partial f^-}{\partial x} + \frac{1}{2\tau}(f^+ - f^-). \tag{17}$$

In terms of the total distribution function $f=f^+ + f^-$ and the current $J=c(f^+ - f^-)$, the equations take the form

$$\frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x}, \quad \frac{\partial J}{\partial t} + \frac{1}{\tau}J = -c^2\frac{\partial f}{\partial x}. \tag{18}$$

This leads immediately to the 1D telegraph equation for f

$$\frac{\partial^2 f}{\partial t^2} + \frac{1}{\tau}\frac{\partial f}{\partial t} = c^2\frac{\partial^2 f}{\partial x^2}, \tag{19}$$

and also for J and each components f^+ and f^-.

The isomorphism between persistent random walk and dissipative wave propagation, though very attractive from many points of view, does not go beyond 1D. It is true that the multidimensional telegraph Eq. (16) follows readily from the generalization of Eq. (18)

$$\frac{\partial f}{\partial t} = -\vec{\nabla}\cdot\vec{J}, \quad \frac{\partial \vec{J}}{\partial t} + \frac{1}{\tau}\vec{J} = -c^2\vec{\nabla}f. \tag{20}$$

However, these generalized equations do not follow merely from multidimensional extension of 1D random walk [Eq. (17)]. For instance, such an extension for 2D has a form

$$\frac{\partial f_x^+}{\partial t} = -c\frac{\partial f_x^+}{\partial x} + \frac{1}{2\tau}(-3f_x^+ + f_x^- + f_y),$$

$$\frac{\partial f_x^-}{\partial t} = +c\frac{\partial f_x^-}{\partial x} + \frac{1}{2\tau}(-3f_x^- + f_x^+ + f_y),$$

$$\frac{\partial f_y^+}{\partial t} = -c\frac{\partial f_y^+}{\partial y} + \frac{1}{2\tau}(-3f_y^+ + f_y^- + f_x),$$

$$\frac{\partial f_y^-}{\partial t} = +c\frac{\partial f_y^-}{\partial y} + \frac{1}{2\tau}(-3f_y^- + f_y^+ + f_x), \tag{21}$$

where $f_x=f_x^+ + f_x^-$ and $f_y=f_y^+ + f_y^-$. Adding and subtracting lead to the following equations for the partial densities:

$$\frac{\partial f_x}{\partial t} = -\frac{\partial J_x}{\partial x} + \frac{1}{\tau}(-f_x + f_y),$$

$$\frac{\partial f_y}{\partial t} = -\frac{\partial J_y}{\partial y} + \frac{1}{\tau}(-f_y + f_x), \tag{22}$$

and for the current components

$$\frac{\partial J_x}{\partial t} = -c^2\frac{\partial f_x}{\partial x} - \frac{1}{\tau}J_x,$$

$$\frac{\partial J_y}{\partial t} = -c^2\frac{\partial f_y}{\partial y} - \frac{1}{\tau}J_y. \tag{23}$$

Thus, for the total distribution function $f=f_x+f_y$ and the current $\vec{J}=\vec{i}J_x+\vec{j}J_y$ one obtains

$$\frac{\partial f}{\partial t} = -\vec{\nabla}\cdot\vec{J},$$

$$\frac{\partial \vec{J}}{\partial t} + \frac{1}{\tau}\vec{J} = -c^2\left\{\vec{i}\frac{\partial f_x}{\partial x} + \vec{j}\frac{\partial f_y}{\partial y}\right\} \neq -c^2\vec{\nabla}f. \tag{24}$$

This differs from Eqs. (20), and therefore the two-dimensional telegraph equation is not recovered.

Let us modify Eqs. (21) in precisely the same way as in the previous section. Namely, in addition to transitions with isotropic rate $1/2\tau$, let us introduce anisotropic transitions $(x,+)\leftrightarrow(x,-)$ and $(y,+)\leftrightarrow(y,-)$ with the rates g_x and g_y, respectively,

$$\frac{\partial f_x^+}{\partial t} = -c\frac{\partial f_x^+}{\partial x} - g_x + \frac{1}{2\tau}(-3f_x^+ + f_x^- + f_y),$$

$$\frac{\partial f_x^-}{\partial t} = +c\frac{\partial f_x^-}{\partial x} + g_x + \frac{1}{2\tau}(-3f_x^- + f_x^+ + f_y),$$

$$\frac{\partial f_y^+}{\partial t} = -c\frac{\partial f_y^+}{\partial y} - g_y + \frac{1}{2\tau}(-3f_y^+ + f_y^- + f_x),$$

$$\frac{\partial f_y^-}{\partial t} = +c\frac{\partial f_y^-}{\partial y} + g_y + \frac{1}{2\tau}(-3f_y^- + f_y^+ + f_x) \tag{25}$$

with g_α given by Eq. (12),

$$g_x = \frac{c}{2}\frac{\partial f_y}{\partial x}, \quad g_y = \frac{c}{2}\frac{\partial f_x}{\partial y}. \tag{26}$$

In this case Eqs. (22) for $\partial f_\alpha/\partial t$ do not change, while Eqs. (23) for $\partial J_\alpha/\partial t$ take the form

$$\frac{\partial J_x}{\partial t} + \frac{1}{\tau}J_x = -c^2\frac{\partial f}{\partial x},$$

$$\frac{\partial J_y}{\partial t} + \frac{1}{\tau}J_y = -c^2\frac{\partial f}{\partial y}. \tag{27}$$

Thus both Eqs. (20) are satisfied, from where the two-dimensional telegraph Eq. (16) for f follows immediately.

The generalization of the system (25) for 3D is straightforward, i.e the equation for $f_x^{\pm}$ takes the form

$$\frac{\partial f_x^{\pm}}{\partial t} = \mp c\frac{\partial f_x^{\pm}}{\partial x} \mp g_x + \frac{1}{2\tau}(-5f_x^{\pm} + f_x^{\mp} + f_y + f_z),$$

with $f_z = f_z^+ + f_z^-$ and g_x given by Eq. (15).

IV. KLEIN-GORDON EQUATION

The combination of a substitution $f = \exp(-t/2\tau)\psi$ and analytic continuation of the transition rate to the imaginary value $1/2\tau \to -i\lambda$ converts the telegraph Eq. (16) into the Klein-Gordon equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \Delta \psi - \lambda^2 \psi. \tag{28}$$

This observation was exploited in many works discussing the possibility of the connection between random walks and quantum mechanics. Yet the concept of a complex transition rate does not look particularly attractive.

An alternative stochastic model leading to the Klein-Gordon equation in 1D without recourse to analytic continuation was discussed in [11]. In this model transitions between two velocity states are governed by a "guiding field" $E(x,t)$ which is coupled to the total particle distribution $f(x,t)$ via the Poisson equation. Using the same method as in the previous sections, the model can be extended for 2D as follows:

$$\frac{\partial f_x^+}{\partial t} = -c\frac{\partial f_x^+}{\partial x} - g_x + \frac{1}{2}aE_x,$$

$$\frac{\partial f_x^-}{\partial t} = +c\frac{\partial f_x^-}{\partial x} + g_x - \frac{1}{2}aE_x,$$

$$\frac{\partial f_y^+}{\partial t} = -c\frac{\partial f_y^+}{\partial y} - g_y + \frac{1}{2}aE_y,$$

$$\frac{\partial f_y^-}{\partial t} = +c\frac{\partial f_y^-}{\partial y} + g_y - \frac{1}{2}aE_y. \tag{29}$$

Here the transition rates g_α are given by Eq. (12),

$$g_x = \frac{c}{2}\frac{\partial f_y}{\partial x}, \quad g_y = \frac{c}{2}\frac{\partial f_x}{\partial y}, \tag{30}$$

the field $\vec{E} = \vec{i}E_x + \vec{j}E_y$ satisfies the Poisson equation

$$\vec{\nabla} \cdot \vec{E} = bf, \tag{31}$$

a and b are constants. For the total distribution $f = f_x + f_y = f_x^+ + f_x^- + f_y^+ + f_y^-$ and the current $\vec{J} = c(f_x^+ - f_x^-)\vec{i} + c(f_y^+ - f_y^-)\vec{j}$ one obtains from Eqs. (29)

$$\frac{\partial f}{\partial t} = -\vec{\nabla} \cdot \vec{J},$$

$$\frac{\partial \vec{J}}{\partial t} = -c^2 \vec{\nabla} f + ac\vec{E}. \tag{32}$$

Together with Eq. (31), these equations give the Klein-Gordon equation (28) for f with $\lambda^2 = abc$. The possibility to interpret this approach in the spirit of the Dirac sea model was discussed in [11].

The generalization of this scheme for 3D is straightforward and involves transition rates g_α in the form (15). For instance, the equations for $f_x^{\pm}$ take the form

$$\frac{\partial f_x^+}{\partial t} = -c\frac{\partial f_x^+}{\partial x} - \frac{c}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_z}{\partial x}\right) + \frac{1}{2}aE_x,$$

$$\frac{\partial f_x^-}{\partial t} = +c\frac{\partial f_x^-}{\partial x} + \frac{c}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_z}{\partial x}\right) - \frac{1}{2}aE_x, \tag{33}$$

which gives

$$\frac{\partial f_x}{\partial t} = -\frac{\partial J_x}{\partial x},$$

$$\frac{\partial J_x}{\partial t} = -c^2\frac{\partial f}{\partial x} + acE_x. \tag{34}$$

These and similar equations for y and z components lead to Eq. (32) and therefore to the Klein-Gordon equations in 3D. Note that if $\vec{E}$ is a conservative vector field, $\vec{\nabla} \times \vec{E} = 0$, then it follows from Eq. (32) that the vorticity of the current $\vec{\nabla} \times \vec{J}$ ("spin") is a constant of motion.

V. CONCLUSION

In this paper we constructed systems of partial differential equations of first order which can be interpreted in terms of stochastic motion of an ensemble of particles and lead to wave equations for the particle distribution in any dimensions. There is nothing unusual, of course, in wavelike dynamics in a system of interacting particles. However, the presented model attempts to picture waves not as a dynamical process but kinematically, assuming that the particles move with speed equal to the phase speed of the wave and represent medium excitations rather than medium constituents. Technically, the approach is based on a simple trick of "completing the gradient." Namely, an appropriate 1D single-particle process is extended to higher dimensions supplemented with additional terms g_α, which make the equation for the local current to involve the gradient of the total distribution (e.g., $\partial \vec{J}/\partial t = -c^2 \vec{\nabla} f$ for the wave equation). The additional terms g_α can be interpreted as transition rates between states with opposite velocity directions and make the model consistent with the Huygens' principle. The discreteness of velocity directions, assumed in this approach, may look physically artificial, yet technically it is essential. The model can be readily extended for any finite number of allowed directions, but not for a continuum of directions.

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